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A NOTE ON GLOBAL COTOTAL DOMINATION IN GRAPHS

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ABSTRACT

A dominating set D of a graph G is a global cototal dominating set if D is both a global dominating set and a cototal dominating set. The global cototal domination number $\gamma_{gcot}(G)$ is the minimum cardinality of a global cototal domination set of G. In this paper we determine the value of the global cototal domination number $\gamma_{gcot}(G)$ for Friendship graph, Helm graph, Trestled graph, Total graph, Web graph and Grid graph.
Subject Classification: 05C69

KEYWORDS:

Global domination number, Cototal domination number, Global cototal domination number.

1. INTRODUCTION

All graphs considered in this paper are simple, finite, undirected and connected. For graph theoretical terms we refer Harary [6] and for terms related to domination we refer Haynes et al. [7]. A set of vertices D in a graph G is a dominating set, if each vertex of G is dominated by some vertices of D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set D of a graph G is a global dominating set if D is also a dominating set of $\bar{G}$. The global domination number $\gamma_g(G)$ is the minimum cardinality of a global dominating set of G. This concept was introduced independently by Brigham and Dutton [2] (the term factor domination number was used) and Sampathkumar [11]. A dominating set D of a graph G is a cototal dominating set if the induced sub graph, $\langle V-D \rangle$ has no isolated vertices. The cototal domination number $\gamma_{cot}(G)$ is the minimum cardinality of a cototal dominating set of G. This concept was introduced by Kulli, Janakiram and Iyer in [8]. A dominating set D of a graph G is a global cototal dominating set if D is both a global dominating set and a cototal dominating set. The global cototal domination number $\gamma_{gcot}(G)$ is the minimum cardinality of a global cototal domination set of G. This new concept, the global cototal domination number $\gamma_{gcot}(G)$ of a graph G was introduced by Sheeba Helen and Nicholas in [9]. In this paper we study the minimal condition of a global cototal dominating set and calculate the global cototal domination number in specific classes of graphs.

We need the following.

Proposition 1.1.[9] For any complete graph K_n, $\gamma_{gcot}(K_n) = n$, $n \geq 3$.

Proposition 1.2.[9] For any star graph $K_{1,n}$, $\gamma_{gcot}(K_{1,n}) = n + 1$, $n \geq 3$.

Proposition 1.3.[9] For the cycle C_n, $n \geq 6$

$$\gamma_{gcot}(C_n)= \begin{cases} \frac{n}{3}, & n \equiv 0\ (mod\ 3); \\ \left\lceil\frac{n}{3}\right\rceil, & n \equiv 1\ (mod\ 3); \\ \left\lceil\frac{n}{3}\right\rceil + 1, & n \equiv 2\ (mod\ 3). \end{cases}$$

Proposition 1.4[9]. For any wheel W_n, $\gamma_{gcot}(W_n) = \begin{Bmatrix} 4 & if\ n = 3; \\ 3 & otherwise. \end{Bmatrix}$.

2. MAIN RESULTS

Definition 2.1[9]

A global cototal dominating set of a graph G is a set D of vertices of G such that D is both global dominating set and cototal dominating set. The global cototal domination number $\gamma_{gcot}(G)$ is the minimum cardinality of a global cototal dominating set of G.

The following theorem of Ore characterizes the minimal dominating sets.

Theorem 2.2. [10] A dominating set D is a minimal dominating set if and only if for each vertex v in D one of the following condition holds.

(i) v is an isolated vertex of D.

(ii) There exists a vertex u in V−D such that N(u) ∩ D = {v}.

Theorem 2.3. A global cototal dominating set D is minimal if and only if for each vertex v in D one of the following conditions holds.

(i) There exists a vertex u in V−D such that N(u) ∩ D = {v}.

(ii) N(u) ∩ (V−D) ≠ ∅.

Proof: Suppose D is the minimal global cototal dominating set of G. On contrary if there exists a vertex v in D such that v does not satisfy any of the given conditions, then by the previous theorem, D_1 = D−{v} is a dominating set of G. By sub division(ii) <V−D_1> has no isolated vertices. This implies that D_1 is a global cototal dominating set of G which is a contradiction. Sufficiency is obvious. ■

Theorem 2.4. Let T be the spanning sub graph of a complete graph K_n, then $\gamma_{gcot}(T) \leq \gamma_{gcot}(K_n)$.

Proof: Let D be the minimal global cototal dominating set of K_n. Let T be the spanning sub graph of a K_n. We know that $\gamma_{gcot}(K_n) = n$, $n \geq 3$, by Proposition 1.1. If T is a star then by Proposition 1.2 the global cototal domination number of a star graph of order n is n. Hence equality holds. If T is a cycle, by Proposition 1.3 for $n \geq 6$

$$\gamma_{gcot}(C_n)= \begin{cases} \frac{n}{3}, & n \equiv 0\ (mod\ 3); \\ \left\lceil\frac{n}{3}\right\rceil, & n \equiv 1\ (mod\ 3); \\ \left\lceil\frac{n}{3}\right\rceil + 1, & n \equiv 2\ (mod\ 3). \end{cases}$$

This proves the theorem ■

Definition 2.5. Friendship graph $C_3^{(t)}$ is a planar undirected graph with 2n+1 vertices and 3n edges. It can be got by joining'‘t’ copies of the cycle graph C_3 at a common vertex.

Theorem 2.6. $\gamma_{gcot}(C_3^{(t)}) = 3$ where t denotes the number of copies of the cycle C_3 identified at a common vertex.

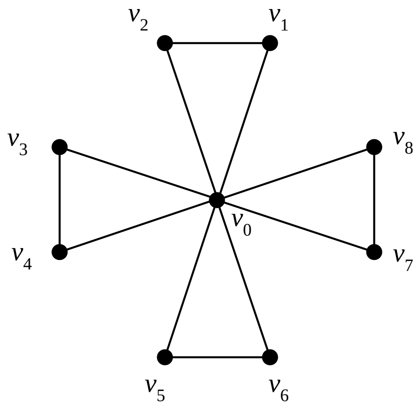

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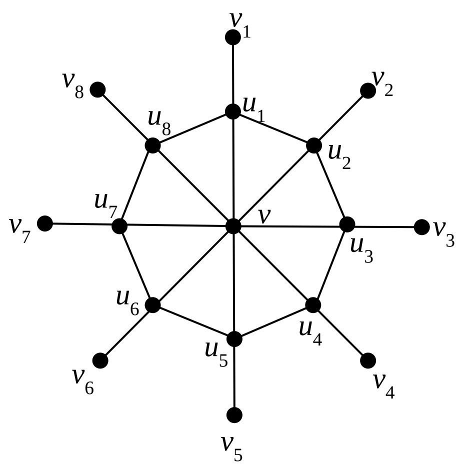
Figure 2.4

Proof: Let D be the minimal global cototal dominating set of $C_3^{(t)}$. v_0 be the apex vertex in $C_3^{(t)}$.Then v_0 dominates all other vertices of $C_3^{(t)}$. Hence any minimal global cototal dominating set D must contain v_0. Since v_0 is isolated in G^c, it does not dominate any vertex in G^c. Therefore D has a vertex $v_1 \neq v_0$. But in this case the sub graph induced by $<V-\{v_1, v_0\}>$ has an isolated vertex v_2 which is adjacent to both v_0 and v_1 in G. Hence $D = \{v_0, v_1, v_2\} \subseteq V(G)$. Now D is the minimal global cototal dominating set of $C_3^{(t)}$ and hence $\gamma_{gcot}(C_3^{(t)}) = 3$. ■

Definition 2.7. A graph obtained from a wheel by attaching a pendant edge at each vertex of an n-cycle is a helm and is denoted by H_n. It is a graph of order 2n+1.

Theorem 2.8. $\gamma_{gcot}(H_n) = n+1$.



Helm Graph H_8

Figure 2.5

Proof: Let $u_1, u_2, u_3, \ldots, u_n$ be the vertices of the cycle. Let $v_1, v_2, v_3, \ldots, v_n$ be the corresponding pendant vertices and v be the center. H_n contains 2n+1 vertices. Let D be the minimal global cototal dominating set. Then D contains v. Since v dominates only the cycle vertices $u_1, u_2, u_3, \ldots, u_n$, D must contain a pendant vertex. Then D must contain all the pendant vertices, $v_1, v_2, v_3, \ldots, v_n \in D$ since otherwise it will violate the cototal property. Hence we claim that $D = \{v, v_1, v_2, v_3, \ldots, v_n\}$ is minimal. The vertex v_i cannot be replaced with the corresponding u_i, since otherwise, $V - \{v, u_1, u_2, u_3, \ldots, u_n\}$ induce isolated vertices $v_1, v_2, v_3, \ldots, v_n$ which violates the cototal property. Therefore we choose $D = \{v_1, v_2, v_3, \ldots, v_n, v\} \subseteq V(G)$ as the minimal global cototal dominating set of H_n and hence $\gamma_{gcot}(H_n) = n+1$. ■

Definition 2.9. The trestled graph of index k denoted by $T_k(G)$ is a graph obtained from G adding k copies of K_2 corresponding to each edge uv of G and joining u and v to the respective end vertices of each K_2.

Theorem 2.10. If G is a trestled graph of index k of a cycle C_n, then $\gamma_{gcot}(G) = n$.

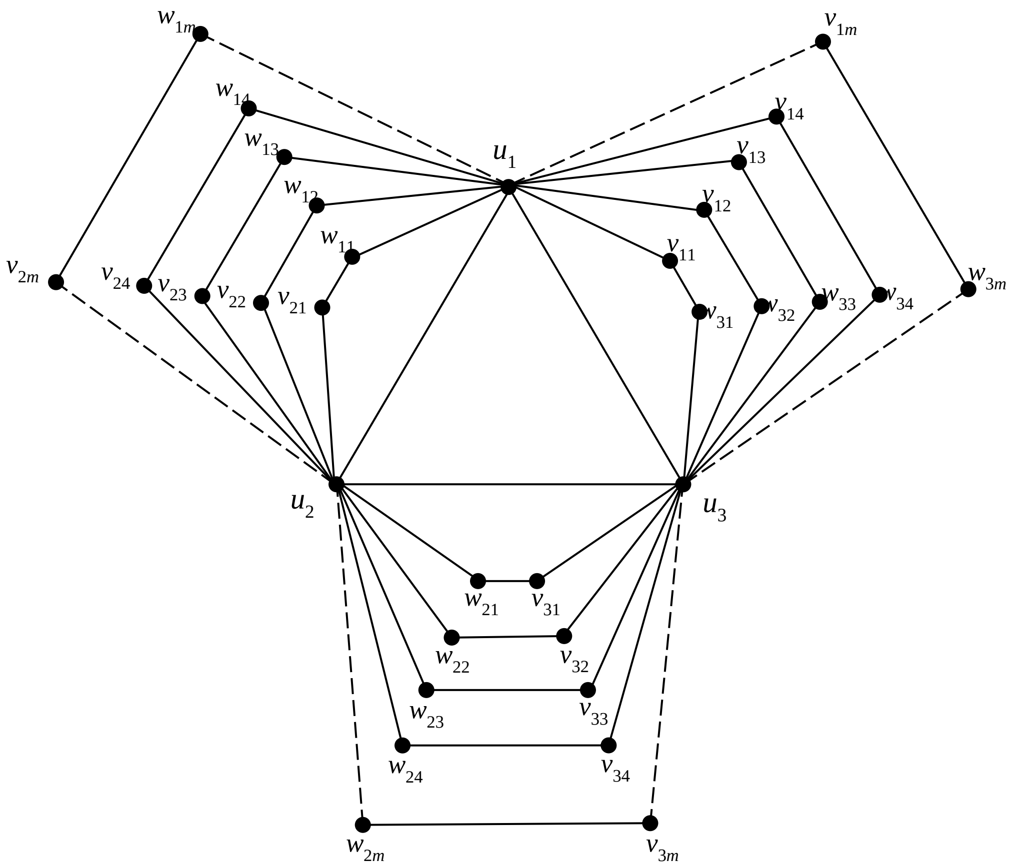


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Figure 2.6

Proof: Let $u_1, u_2, u_3, \ldots, u_n$ be the vertices of the cycle C_n. The k copies of K_2 corresponding to each edge u_1u_2 is labeled as $w_{11}, w_{12}, w_{13}, \ldots, w_{1m}$ and $v_{21}, v_{22}, \ldots, v_{2m}$. In general k copies of K_2 corresponding to each edge u_iu_j $(1 \le i, j \le n)$ is labeled as $w_{i1}, w_{i2}, w_{i3}, \ldots, w_{im}$ and $v_{j1}, v_{j2}, \ldots, v_{jm}$. Let D be the minimal global cototal dominating set of G. The trestled graph of a cycle C_n of index k contains n+2kn vertices. u_i is adjacent to $w_{i1}, w_{i2}, w_{i3},$

. . . , w_{im} and $v_{j1}, v_{j2}, \ldots, v_{jm}$ of k copies of K_2 corresponding to each edge u_iu_j and ui is adjacent to the succeeding and preceding vertices u_{i+1}, u_{i+2} of the cycle. The end vertices of each K_2 is adjacent to the end vertices of each edge u_iu_j of the cycle. Thus $D = \{u_1, u_2, u_3, \ldots, u_n\} \subseteq V(G)$ is minimal and the induced subgraph $< V-D >$ results in a disconnected graph containing nk number of K_2's. Thus D is the minimal global cototal dominating set of G. Hence $\gamma_{gcot}(G) = n$. ■

Corollary 2.11. If $G \cong T_k(C_n)$, then $\gamma_{gcot}(G) = \gamma(G)$.

Theorem 2.12. If $G \cong T_k(P_n)$, then $\gamma_{gcot}(T_k(P_n)) = n$ for every $k \geq 1$.

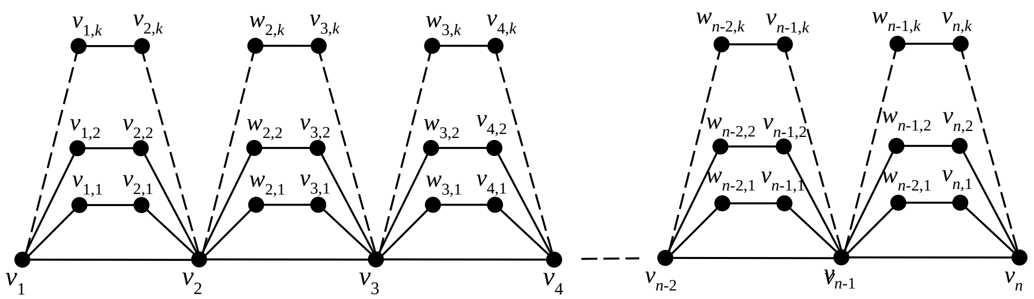

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Figure 2.7

Proof: Let $v_1, v_2, v_3, \ldots, v_n$ denote the vertices of the path P_n. The initial vertex v_1 of the path P_n is adjacent to k vertices namely $v_{11}, v_{12}, v_{13}, \ldots, v_{1k}$ and the end vertex v_n of the path P_n is adjacent to k vertices $v_{n1}, v_{n2}, \ldots, v_{nk}$. Each internal vertex v_i of the path P_n of $T_k(P_n)$ is adjacent to 2k vertices $v_{i1}, v_{i2}, \ldots, v_{ik}, w_{i1}, w_{i2}, \ldots, w_{ik}$. Then $|V(T_k(P_n))| = (n-2)(2k+1) + 2k+2 = (2k+1)n - 2k$. Let D be the minimal global cototal dominating set of $T_k(P_n)$. We have $\Delta(T_k(P_n)) = 2k+2$. Since $\deg(v_i) = 2k+2$, $(2 \leq i \leq n-1)$, $v_i \in D$. The induced sub graph <V−D> has 2k isolated vertices $v_{11}, v_{12}, v_{13}, \ldots, v_{1k}, v_{n1}, v_{n2}, \ldots, v_{nk}$. Therefore $v_1, v_n \in D$. Thus $D = \{v_1, v_2, v_3, \ldots, v_n\}$ is the minimal global cototal dominating set of $T_k(P_n)$ with $|D| = n$. Hence $\gamma_{gcot}(T_k(P_n)) = n$. ■

Theorem 2.13. If $G \cong T_k(K_{1,n})$, then $\gamma_{gcot}(G) = n+1$ for every $k \geq 1$.

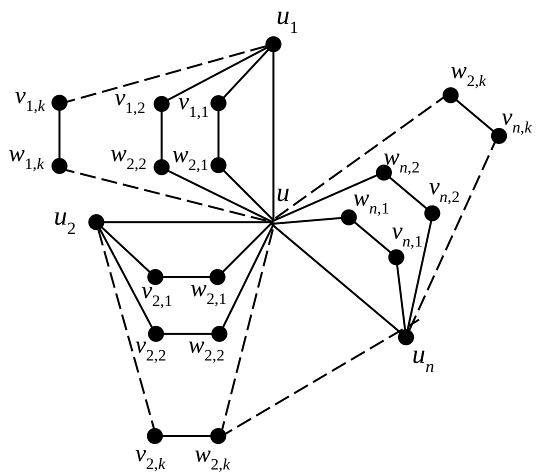

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Figure 2.8

Proof: Let u be the center and $u_1, u_2, u_3, \dots, u_n$ be the pendant vertices of the star $K_{1,n}$. Let $v_{i1} w_{i1}, v_{i2} w_{i2}, \dots, v_{ik} w_{ik}$ be the edges added to the edge uu_i of $K_{1,n}$ as shown in the figure. Let D be the minimal global cototal dominating set of $T_k(K_{1,n})$. The vertex u dominates $u_1, u_2, \dots, u_n$ and one end of the edges $v_{i1}w_{i1}, v_{i2}w_{i2}, \dots, v_{ik} w_{ik}$. Since the degree of u is maximum $u \in D$. the only possible global cototal dominating sets of $T_k(K_{1,n})$ are $D_1 = \{u, v_{11}, v_{12}, v_{13}, \dots, v_{1k}, v_{21}, v_{22}, v_{23}, \dots, v_{2k}, \dots, v_{n1}, v_{n2}, \dots, v_{nk}\}$ with $|D_1| = nk+1$ and $D_2 = \{u, u_1, u_2, u_3, \dots, u_n\}$ with $|D_2| = n+1$. Since $|D_2| < |D_1|$ we take $D_2 = D = \{u, u_1, u_2, u_3, \dots, u_n\}$ as a γ_{gcot}-set of $T_k(K_{1,n})$ with $|D| = n+1$. Also the induced sub graph $\langle V-D \rangle$ is disconnected containing nk number of K_2 graphs. Hence $\gamma_{gcot}(G) = n+1$. ■

Definition 2.14. The total graph T(G) of a graph G = (V, E) has vertices that correspond one to one with elements of $V \cup E$ and two vertices in T(G) are adjacent if and only if the corresponding elements are adjacent or incident in G.

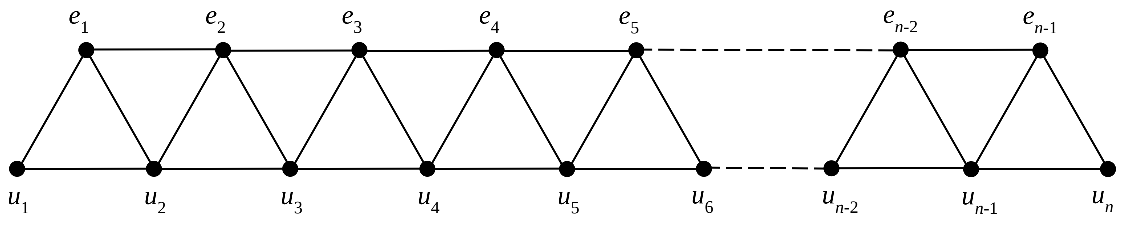

Figure 2.9

Theorem 2.15. If $G \cong T(P_n)$, then $\gamma_{gcot}(G) = \left\lceil \frac{2n-1}{5} \right\rceil$.

Proof: Let $\{u_1, u_2, u_3, \dots, u_n, e_1, e_2, e_3, \dots, e_{n-1}\}$ be the vertex set of $T(P_n)$ with 2n-1 vertices. Let D_1 be the global cototal dominating set of $T(P_n)$ containing vertices u_i, $1 \le i \le n$. Let D_2 be the global cototal dominating set of $T(P_n)$ containing vertices e_i, $1 \le i \le n-1$. Let D_3 be the minimal global cototal dominating set of $T(P_n)$. Now let us choose the first five vertices of $T(P_n)$ say e_1, e_2, u_1, u_2, u_3 here u_2 is adjacent to all the vertices of the above set. Hence $u_2 \in D_3$. Similarly for the set of five vertices $e_i, e_{i+1}, u_i, u_{i+1}, u_{i+2}$ here u_{i+1} dominates all the other vertices. Hence $u_{i+1} \in D_3$.Next we choose the proceeding five vertices of $T(P_n)$ say e_3, e_4, e_5, u_4, u_5. The vertex e_4 dominates all the other vertices. Hence $e_4 \in D_3$. For the next set of five vertices $e_i, e_{i+1}, e_{i+2}, u_{i+1}, u_{i+2}$ here e_{i+1} dominates all the other vertices in the set. Hence $e_{i+1} \in D_3$. Proceeding like this we have the remaining vertices as i = 0, 1, 2, 3, 4. Among these vertices, choose a vertex which is having the maximum degree. Finally we have $D_3 < D_2 < D_1$ with $|D_3| = \left\lceil \frac{2n-1}{5} \right\rceil$. D_3 is a minimal global cototal dominating set with $|D_3| = \left\lceil \frac{2n-1}{5} \right\rceil$. Hence $\gamma_{gcot}(G) = \left\lceil \frac{2n-1}{5} \right\rceil$. ■

Definition 2.16. The web graph is a graph obtained by joining the pendant vertices of a Helm H_n to form a cycle and then adding a single pendant edge to each vertex of this outer cycle. It is a graph of order 3n+1.

Theorem 2.17. For a web graph G, $\gamma_{gcot}(G) = n+1$.

Proof: $V(G) = \{u, u_i, v_i, w_i \,/\, i = 1, 2, \dots, n\}$ and $E(G) = \{uu_i, u_iv_i, v_iw_i \,/\, i = 1, 2, \dots, n\} \cup \{u_i u_{i+1}, v_iv_{i+1} \,/\, i = 1, 2, \dots, n-1\} \cup \{u_nu_1, v_nv_1\}$. Let D be the minimal global cototal dominating set. Then D contains u, since u dominates only the cycle $u_1, u_2, \dots, u_n$. D must contain a pendant vertex. Then D must contain all the pendant vertices $w_1, w_2, \dots, w_n$ which dominates the vertices of the outer cycle, since otherwise, it will violate the cototal property.

That is $w_1, w_2, \ldots, w_n \in D$. We claim that $\{u, w_1, w_2, \ldots, w_n\}$ is minimal. The vertex w_i cannot be replaced with the corresponding v_i, since otherwise $V - \{u, v_1, v_2, \ldots v_n\}$ induce a cycle C_n and isolated vertices $w_1, w_2, \ldots, w_n$ which violates the cototal property. Therefore we choose $D = \{u, w_1, w_2, \ldots, w_n\} \subseteq V(G)$ as the minimal global cototal dominating set of the web graph and hence $\gamma_{gcot}(G) = n+1$. ■

Definition 2.18. The Grid graph P_m x P_n is the Cartesian product of two paths P_m and P_n

Theorem 2.19. The global cototal domination number of a grid graph is given by

$$\gamma_{gcot}(P_m \text{ x } P_n) = \begin{cases} \frac{mn}{4} + \frac{m}{2} & if\ m\ is\ even, n\ is\ even; \\ \left\lceil \frac{mn}{4} \right\rceil + \left\lceil \frac{m}{2} \right\rceil & if\ m\ is\ odd, n\ is\ even; \\ \left\lceil \frac{mn}{5} \right\rceil + \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil & if\ m\ is\ odd, n\ is\ odd; \\ \frac{m}{2} \left\lceil \frac{n}{2} \right\rceil & if\ m\ is\ even\ n\ is\ odd. \end{cases}$$

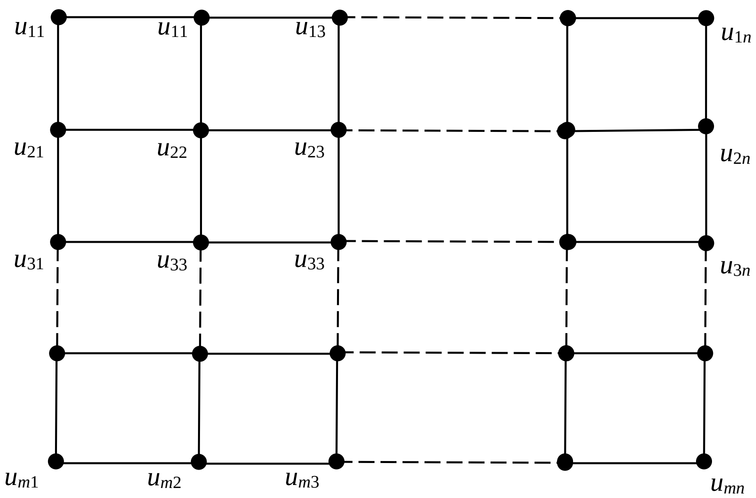

Figure 2.11

Proof: Let $V(P_m \text{ x } P_n) = \{u_{ij} / i = 1, 2, 3, \ldots, m, j = 1, 2, 3, \ldots, n\}$

Consider the following sets

$D_1 = \bigcup_{t=1}^{m/2} \left\{ \bigcup_{s=1}^{\lceil n/4 \rceil} U_{(2t-1)(4s-3)} \bigcup_{s=1}^{\lfloor n/4 \rfloor} U_{2t,(4s-1)} \right\} \bigcup_{t=1}^{m/2} U_{2t,n}$, where m is even and n is even.

$D_2 = \bigcup_{t=1}^{(m+1)/2} \left\{ \bigcup_{s=1}^{\lceil n/4 \rceil} U_{(2t-1)(4s-3)} \bigcup_{s=1}^{\lfloor n/4 \rfloor} U_{2t,(4s-1)} \right\} \bigcup_{t=1}^{(m-1)/2} U_{2t,n}$, where m is odd and n is even.

$D_3 = \bigcup_{t=1}^{m/2} \left\{ \bigcup_{s=1}^{\lceil n/4 \rceil} U_{(2t-1)(4s-3)} \bigcup_{s=1}^{\lfloor n/4 \rfloor} U_{2t,(4s-1)} \right\}$, where m is even and n is odd.

$D_4 = \bigcup_{t=1}^{(m+1)/2} \left\{ \bigcup_{s=1}^{\lceil n/4 \rceil} U_{(2t-1)(4s-3)} \bigcup_{s=1}^{\lfloor n/4 \rfloor} U_{2t,(4s-1)} \right\}$, where m is odd and n is odd.

Consider a 4- square centered at $u_{ij} \in D$ contains 9 vertices namely $u_{i-1,j-1}, u_{i-1,j}, u_{i-1,j+1}, u_{i,j-1}, u_{i,j}, u_{i,j+1}, u_{i+1,j-1}, u_{i+1,j}, u_{i+1,j+1}$. Then u_{ij} dominates the vertices of $N(u_{ij}) = \{u_{i,j-1}, u_{i,j+1}, u_{i-1,j}, u_{i+1,j}\}$. The vertices $u_{i-1,j-1}, u_{i-1,j+1}, u_{i+1,j-1}, u_{i+1,j+1}$ are dominated by $u_{i-1,j-2}, u_{i-1,j+2}, u_{i+1,j-2}, u_{i+1,j+2}$ respectively. Hence D is a dominating set of P_m x P_n. Obviously D dominates G^c. Moreover $V(G)-D$ induces a subgraph containing paths P_m's and P_3's and no isolated vertices. Hence D is a global cototal dominating set of P_m x P_n. Let $x \in D$. Suppose we remove x from D the vertex x is not even dominated by any of the vertices of D since D is an independent set.

Hence $|D_1| = \frac{mn}{4} + \frac{m}{2}$, if m is even, n is even.

$|D_2| = \left[\frac{mn}{4}\right] + \left\lceil\frac{m}{2}\right\rceil$, if m is odd, n is even.

$|D_3| = \frac{m}{2}\left\lceil\frac{n}{2}\right\rceil$, if m is even, n is odd.

$|D_4| = \left\lceil\frac{mn}{5}\right\rceil + \left[\frac{m}{3}\right] + \left[\frac{n}{3}\right]$, if m is odd, n is odd. ■

REFERENCES

[1] B.Basavanagoud and S.M Hosamani – Journal of Scientific Research J.Sci.Res3 (3), 547 – 555 (2011)

[2] R.C.Brigham and R.D.Dutton, Factor domination in graphs, Discrete Math 86(1990)127-136.

[3] C.J .Cockayne, R.M .Dawes and S.T. Hedetniemi, Total Domination in graphs, networks, 10(1980) 211-219

[4] Dr.J.Deva Raj,V.Sujin Flower, A note on Global Total Domination in Graphs, Bulletin of Pure and Applied Sciences Volume 30 E (Math & Stat)Issue (No:1) 2011 P.63 - 70

[5] G.S.Domke, J.H.Hattingh, S.T. Hedetniemi, R.C.Laskar and L.R .Markus, Restrained Domination in graphs, Discrete Math 203 (1999).

[6] Harary.F, Graph Theory, Addison-Wesley, Reading, MA, 1972.

[7] T.W. Haynes, S.T. Hedetneimi, P.J. Slater, Fundamentals of Domination in Graphs, MarcelDekker, New York,1988.

[8] V.R.Kulli, B. Janakiram and R.R Iyer. The Cototal Domination Number of a graph J.Discrete Mathematical Sciences and Cryptography 2 (1999) 179-184.

[9] T. Nicholas and T. Sheeba Helen, Global Cototal Domination in Graphs, Tamkang Journal of Mathematics, Communicated.

[10] Ore.O, Theory of Graphs, Amer, Math. Soc. Colloq. Publ., 38, Providence, (1962).

[11] E. Sampathkumar, The global domination number of a graph J.MathPhys.Sci 23 (1989) 377-385.